Spatially-varying meshless approximation method for enhanced computational efficiency

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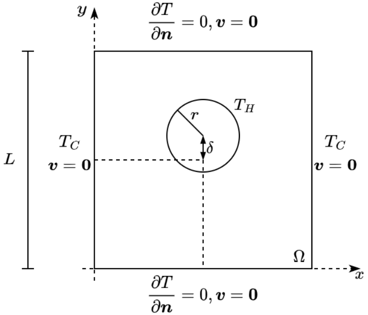
Motivation

Why numerical solution?

Realistic problems do not have closed form solutions.

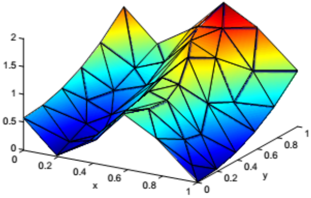
$$\nabla \cdot \vec{v} = 0,$$
$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla p + \nabla \cdot (\mathrm{Ra} \nabla \vec{v}) - \vec{g} \mathrm{Ra} \mathrm{Pr} T_\Delta,$$
$$\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = \nabla \cdot (\nabla T)$$

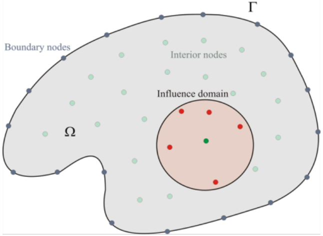


Numerical treatment

Numerical treatment is required:

1. Domain discretization
2. Differential operator approximation
3. PDE discretization
4. Solve sparse linear system





Differential operator approximation

$$(\mathcal{L}u)(\boldsymbol{x}_c) \approx \sum_{i=1}^{n} w_i u(\boldsymbol{x}_i)$$

$$\mathcal{L}\Big|_{\boldsymbol{x}_c} = \boldsymbol{w}_{\mathcal{L}}(\boldsymbol{x}_c)^T$$

Approximation basis

RBF-FD

- Polyharmonic splines augmented with monomials

+ Higher stability
+ Stable on scattered nodes
- Computationally demanding

Stencil size: 12 in 2D, 20 in 3D

MON

- A set of monomials centred at stencil nodes.

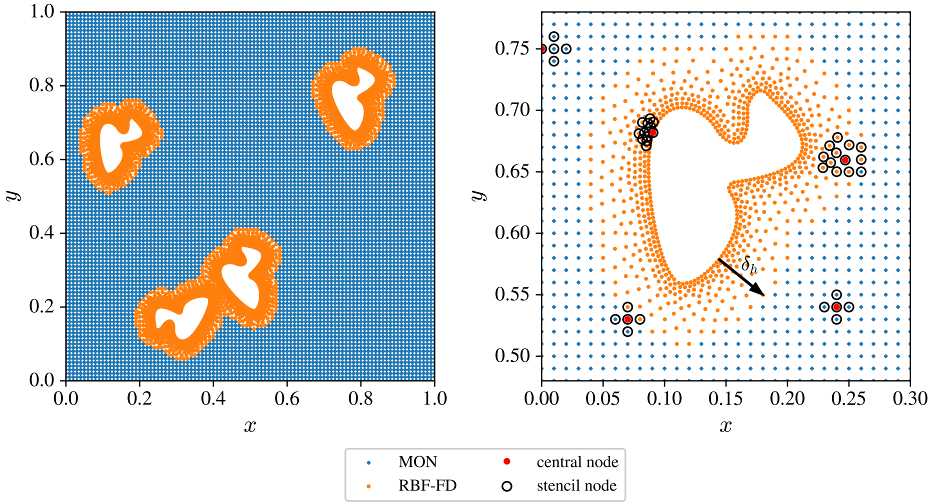
+ Computationally cheap
- Unstable on irregular nodes

Stencil size: 5 in 2D and 7 in 3D

Both allow control over the order of the approximation.

Spatially-varying approximation method

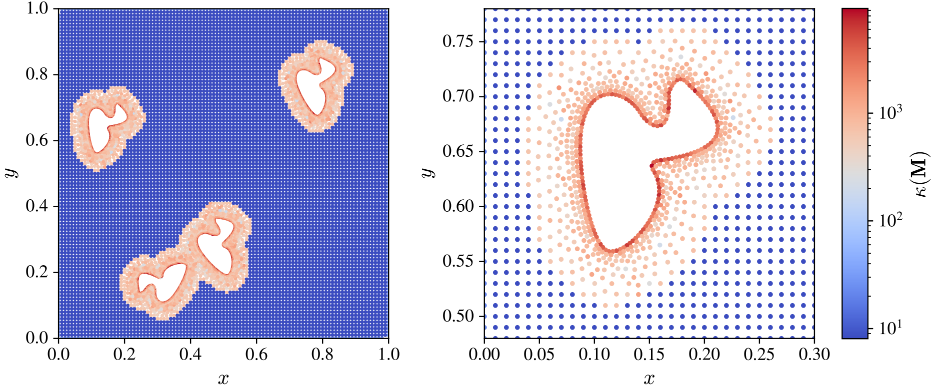
Spatially-varying approximation method



Computational stability

Computational stability

The stability is estimated via the condition number $\kappa(\mathbf{M}) = \|\mathbf{M}\| \left\|\mathbf{M}^{-1}\right\|$, where $\|\cdot\|$ denotes the L^2 norm.



Governing problem

Governing problem

To assess the advantages of the hybrid method, we focus on the natural convection problem that is governed by a system of three PDEs that describe the continuity of mass, the conservation of momentum and the transfer of heat

$$\nabla \cdot \vec{v} = 0, \tag{1}$$

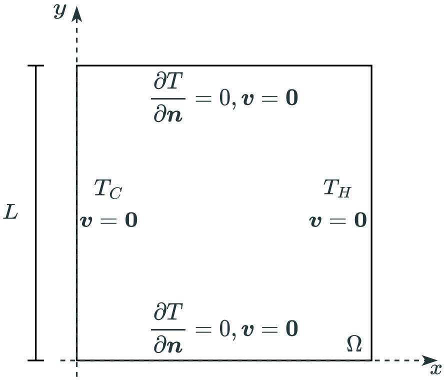
$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla p + \nabla \cdot (\mathrm{Pr} \nabla \vec{v}) - \mathrm{Ra}\mathrm{Pr}\vec{g} T_\Delta, \tag{2}$$

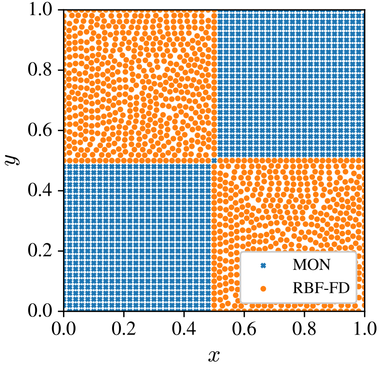
$$\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = \nabla \cdot (\nabla T), \tag{3}$$

where a dimensionless nomenclature using Rayleigh (Ra) and Prandtl (Pr) numbers is used.

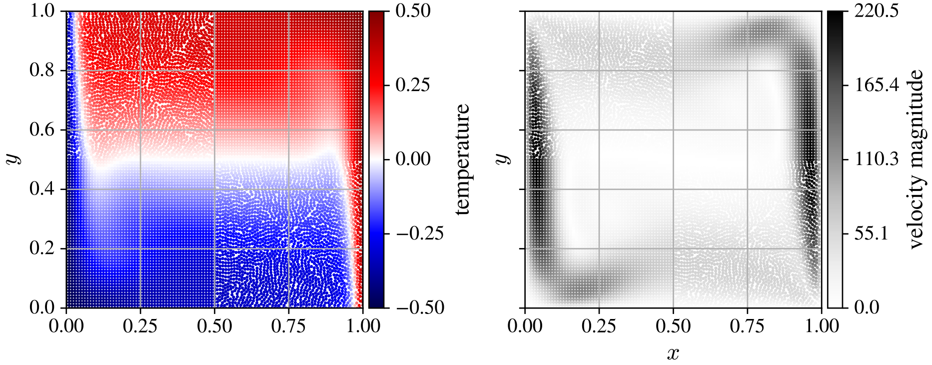
The de Vahl Davis problem

- To establish confidence in the presented solution procedure.
- Because the regularity of the domain shape allows us to efficiently discretize it using exclusively scattered or regular nodes.



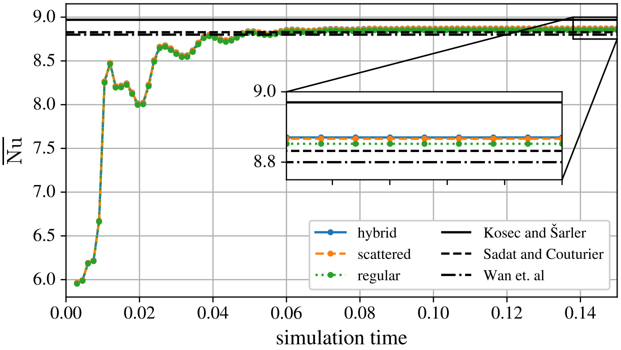


The de Vahl Davis problem: example solution



Nusselt number: a convenient scalar value for comparison with reference solutions. The average Nusselt number ($\overline{\mathrm{Nu}}$) is calculated as the average of the Nusselt values at the cold wall nodes

$$\mathrm{Nu} = \frac{L}{T_H - T_C} \left| \frac{\partial T}{\partial \boldsymbol{n}} \right|_{x=0} . \tag{4}$$



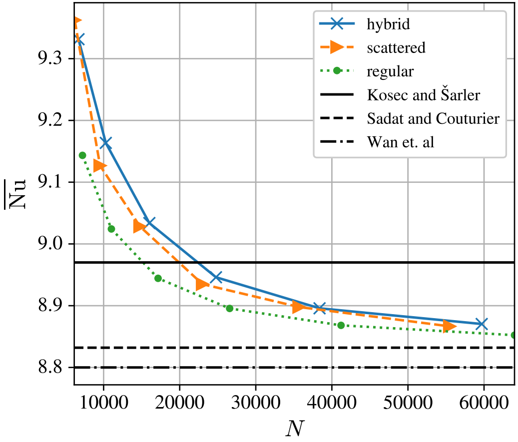
The de Vahl Davis problem: Comparing with references

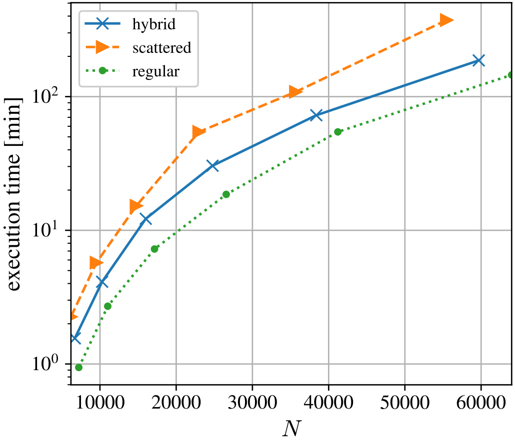
Approximation	$\overline{\mathrm{Nu}}$	execution time [h]	N
scattered	8.867	6.23	55 477
regular	8.852	2.42	64 005
hybrid	8.870	3.11	59 694
Kosec and Šarler (2007)	8.97	/	10201
Sadat and Couturier (2000)	8.828	/	22801
Wan et. al. (2001)	8.8	/	10201

Table 1: Average Nusselt along the cold edge along with execution times and number of discretization nodes.

- The hybrid method shows significantly shorter computational time (about 50 %), than that required by the scattered discretization employing RBF-FD.

The de Vahl Davis problem: Convergence analysis

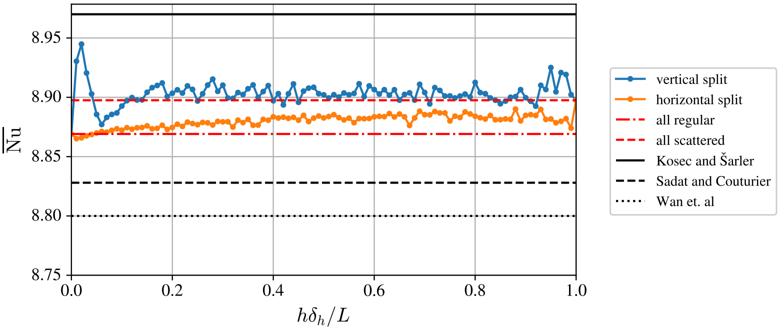




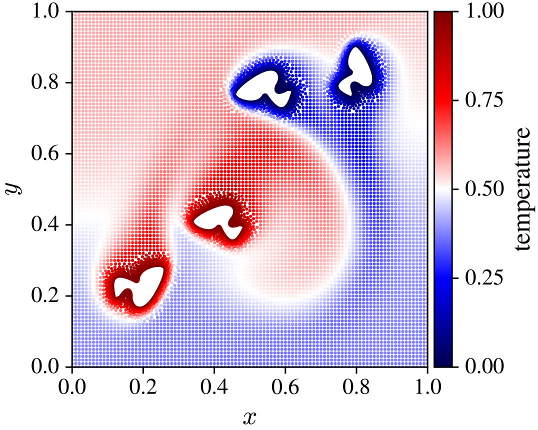
The de Vahl Davis problem: Width of scattered layer δ_h

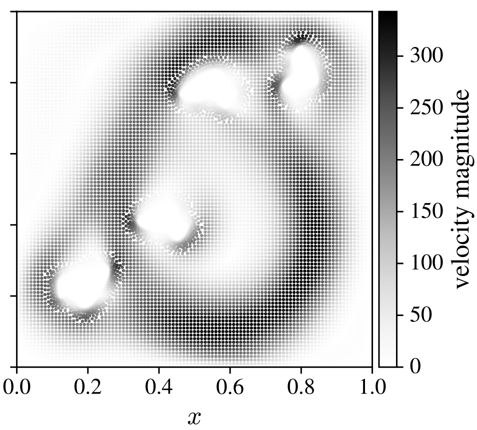
The domain is split into two parts at a distance $h\delta_h$ from the origin in the lower left corner.

- **Horizontal split**, resulting in scattered nodes below the imaginary split and regular nodes above it.
- **Vertical split**, resulting in scattered nodes to the left of it and regular nodes to the right of it.

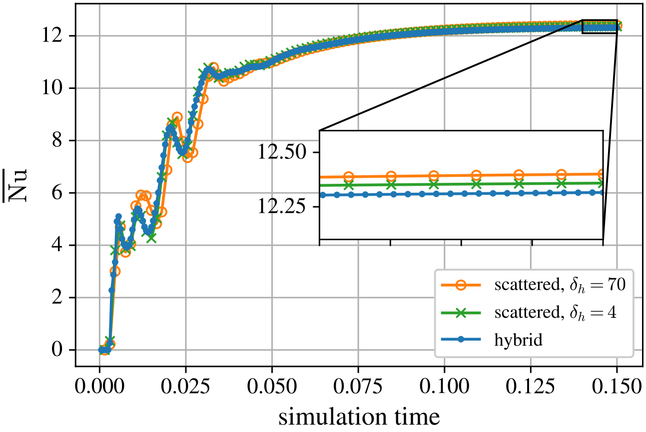


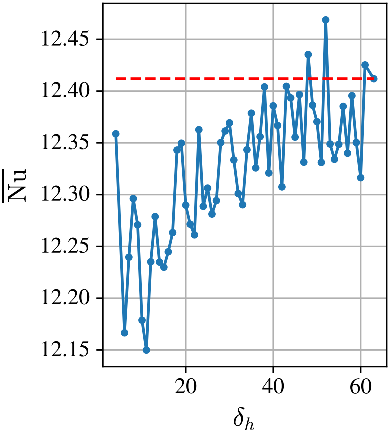
Natural convection on irregularly shaped domains





Natural convection on irregularly shaped domains: Nusselt number





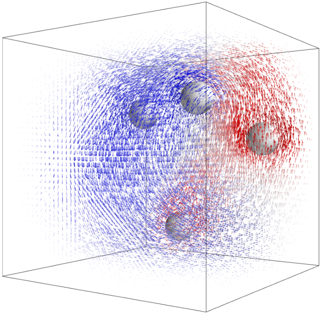
Natural convection on irregularly shaped domains: Computational times

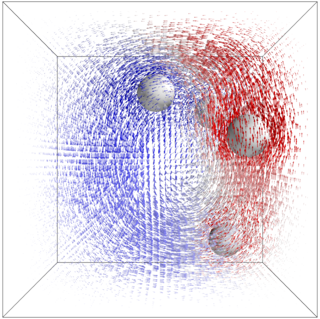
The hybrid method effectively reduces the execution time for approximately 35 %. The pure regular discretization with MON approximation is omitted from the table because a stable numerical solution could not be obtained.

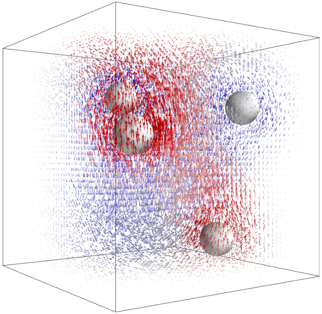
Approximation	$\overline{\mathrm{Nu}}$	execution time [min]	N
scattered	12.32	46.31	10 534
hybrid	12.36	29.11	11 535

Table 2: Average Nusselt along the cold duck edges along with execution times. Note that all values in the table were obtained for $\delta_h = 4$.

Application to three-dimensional irregular domains







Application to three-dimensional irregular domains: Execution times

The scattered method took about 48 hours and the hybrid scattered-regular approximation method took 20 hours to simulate 1 dimensionless time unit with the dimensionless time step $\mathrm{d}t = 7.8125 \cdot 10^{-6}$ and about 75 000 computational nodes with $\delta_h = 4$.

Approximation	$\overline{\mathrm{Nu}}$	execution time [h]	N
scattered	7.36	48.12	65 526
hybrid	6.91	20.54	74 137

Table 3: Average Nusselt along the cold spheres, execution time, and number of computational nodes.

Note: The pure regular discretization with MON approximation is again omitted from the table because a stable numerical solution could not be obtained.

Conclusions

Summary

- Proposed a computationally efficient approach to the numerical treatment of PDEs.
- Verified on a solution to a two-dimensional de Vahl Davis natural convection problem.
- Demonstrated on a solution within irregular two- and three-dimensional domains.

Future work:

- ⋆ A detailed study of the width of the scattered layer δ_h.
- ⋆ Further study of aggressiveness of h-refinement.
- ⋆ Demonstration on more difficult problems: Mixed convection.

Thank you for your attention.

Questions?