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Meshless Adaptive Solution Procedure for Efficient Solving of Partial Differential Equations

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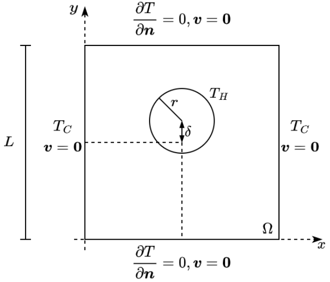
Numerical Treatment of PDEs

1. Domain discretization
2. Differential operator approximation
3. PDE discretization
4. Solve sparse linear system

Meshless approximation:

$$(\mathcal{L}u)(\boldsymbol{x}_c) \approx \sum_{i=1}^{n} w_i u(\boldsymbol{x}_i)$$

Example convection-driven fluid flow problem:



$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \boldsymbol{v} + \frac{1}{\rho}\boldsymbol{b},$$

$$\nabla \cdot \boldsymbol{v} = 0,$$

$$\boldsymbol{b} = \rho(1 - \beta(T - T_{\text{ref}}))\boldsymbol{g},$$

$$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T = \frac{\lambda}{\rho c_p}\nabla^2 T,$$

Approximation Methods

- **Radial Basis Function-generated Finite Differences (RBF-FD)**
  - Polyharmonic Splines augmented with monomials
  - Relatively large support size $n = 2\binom{m+d}{d}$.
- **Diffuse Approximation Method (DAM)**
  - Referred to as Weighted Least Squares (WLS) method
  - Only monomials (less basis functions)
  - Relatively large support size $n = 2\binom{m+d}{d}$
- **The simplest collocation form (MON)**
  - Monomials
  - Small support size $n = 5$ in 2D and $n = 7$ in 3D.
  - Stable only on regular nodes

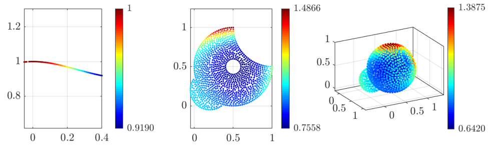
Monomial Augmentation: Problem Setup

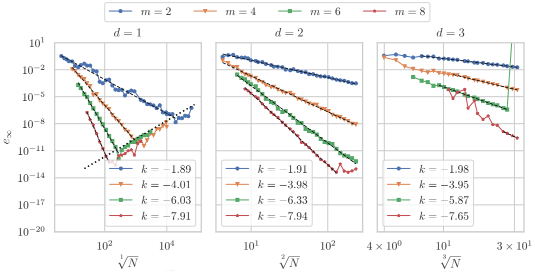
Numerical solution u_h of Poisson's equation with both Dirichlet and Neumann boundary conditions is studied:

$$\nabla^2 u(\boldsymbol{x}) = f_{lap}(\boldsymbol{x}) \quad \text{in } \Omega, \quad (1)$$

$$u(\boldsymbol{x}) = f(\boldsymbol{x}) \quad \text{on } \Gamma_d, \quad (2)$$

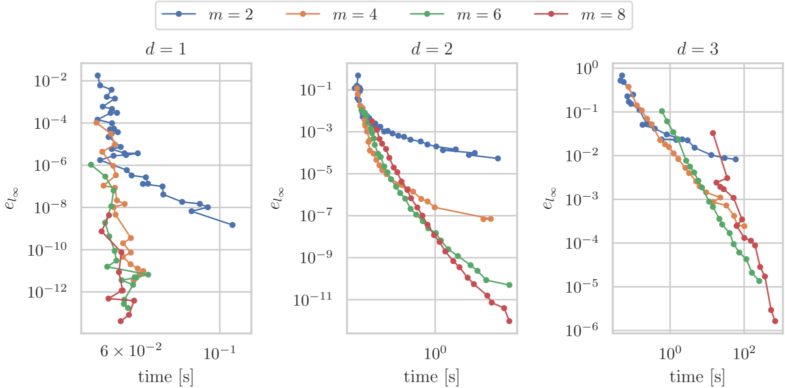
$$\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} = f_{grad}(\boldsymbol{x}) \quad \text{on } \Gamma_n. \quad (3)$$





- Approximation order controlled with the highest order of augmenting monomial.
- Note: recommended stencil size $n = 2\binom{m+d}{d}$

Monomial Augmentation: Time vs. Error



The recommended augmentation order

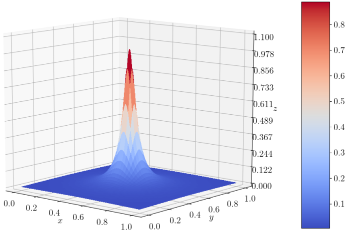
$$m = \frac{5}{4}k + \frac{4}{5}d - 2$$

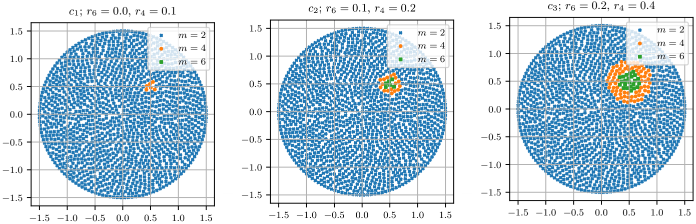
p-refinement

Poisson problem with strong source in the domain

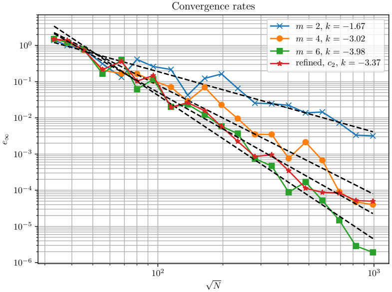
$$\nabla^2 u(\boldsymbol{x}) = f_{\text{lap}}(\boldsymbol{x}), \text{ where}$$

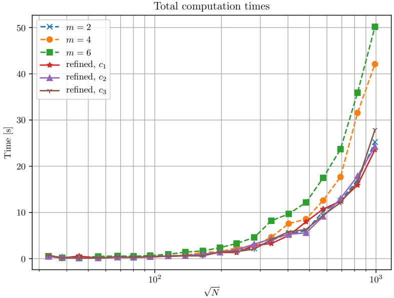
$$f_{\text{lap}}(\boldsymbol{x}) = 3200 \frac{25 \left\|4\boldsymbol{x} - \boldsymbol{2}\right\|^2}{f(\boldsymbol{x})^3} - 800 \frac{d}{f(\boldsymbol{x})}$$





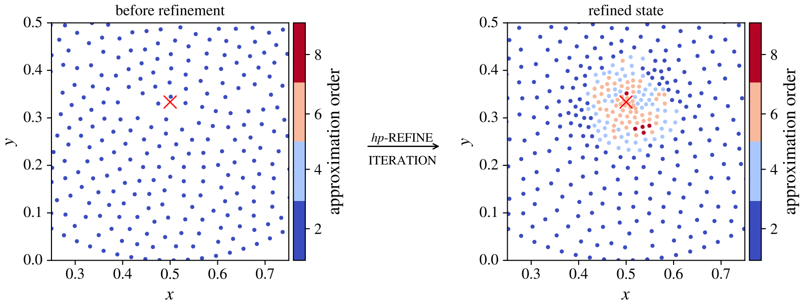
p-refinement: Results





Computational time can be reduced by approximately 50 %. At the same time, accuracy of the numerical solution is notably better compared to unrefined solutions (at second order approximation).

hp-refinement: Goal



Workflow

Based on the well established **solve-estimate-mark-refine** paradigm.

hp-refinement: **solve**-estimate-mark-refine

Poisson problem with exponentially strong source in the domain

$$\nabla^2 u(\boldsymbol{x}) = 2ae^{-a\|\boldsymbol{x}-\boldsymbol{x}_s\|^2}(2a\|\boldsymbol{x}-\boldsymbol{x}_s\| - d) \quad \text{in } \Omega,$$
$$u(\boldsymbol{x}) = e^{-a\|\boldsymbol{x}-\boldsymbol{x}_s\|^2} \quad \text{on } \Gamma_d,$$
$$\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}} = -2a(\boldsymbol{x}-\boldsymbol{x}_s)\boldsymbol{n}e^{-a\|\boldsymbol{x}-\boldsymbol{x}_s\|^2} \quad \text{on } \Gamma_n$$

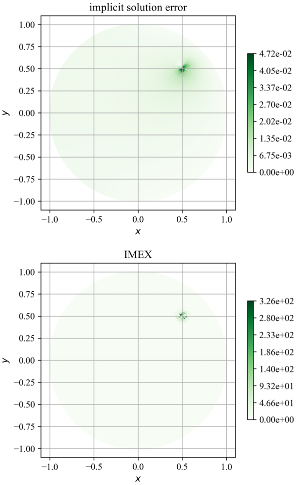
Setup

- RBF-FD
- PHS order $k = 3$
- Monomial augmentation with $m \in \{2, 4, 6, 8\}$
- IMEX with monomials $m \in \{4, 6, 8, 10\}$

hp-refinement: solve-**estimate**-mark-refine

Consider a problem of type $\mathcal{L}u = f_{RHS}$.
The IMplicit-EXplicit error indicator:

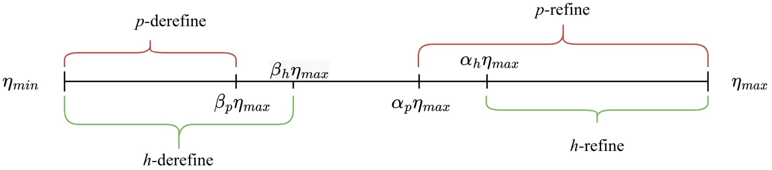
1. Obtain implicit solution $u^{(im)}$ to governing problem using low-order approximations of $\mathcal{L}$, i.e. $\mathcal{L}^{(lo)}_{(im)}$.
2. Obtain high-order approximations of explicit operators $\mathcal{L}$, i.e. $\mathcal{L}^{(hi)}_{(ex)}$
3. Apply $\mathcal{L}^{(hi)}_{(ex)}$ to $u^{(im)}$ and obtain $f_{(ex)}$ in the process
4. Compare f_{RHS} and $f_{(ex)}$



hp-refinement: solve-estimate-**mark**-refine

The modified Texas Three-Fold strategy for error indicator field η

$$
\begin{cases}
\eta_i > \alpha\eta_{max}, & \text{refine} \\
\beta\eta_{max} \le \eta_i \le \alpha\eta_{max}, & \text{do nothing} \\
\eta_i < \beta\eta_{max}, & \text{derefine}
\end{cases}.
$$



Advantage

Easy to understand and implement.

Problem

Does not lead to optimal results.

hp-refinement: solve-estimate-mark-**refine**

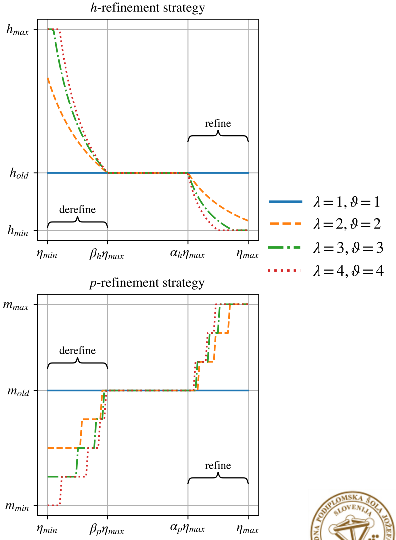
Defining the amount of (de)refinement.

h-refine

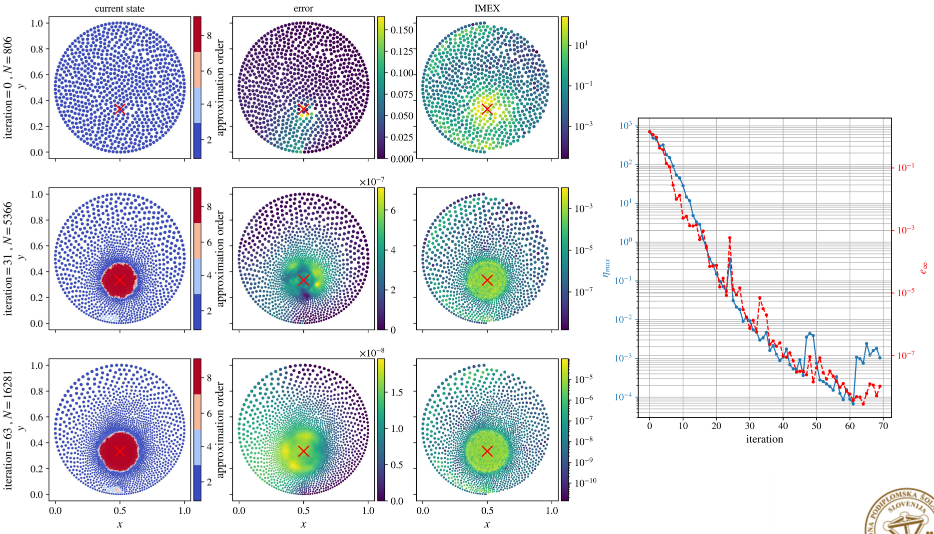
$$h_i^{new}(\boldsymbol{p}) = \frac{h_i^{old}}{\frac{\eta_i - \alpha\eta_{max}}{\eta_{max} - \alpha\eta_{max}}\left(\lambda - 1\right) + 1}$$

h-derefine

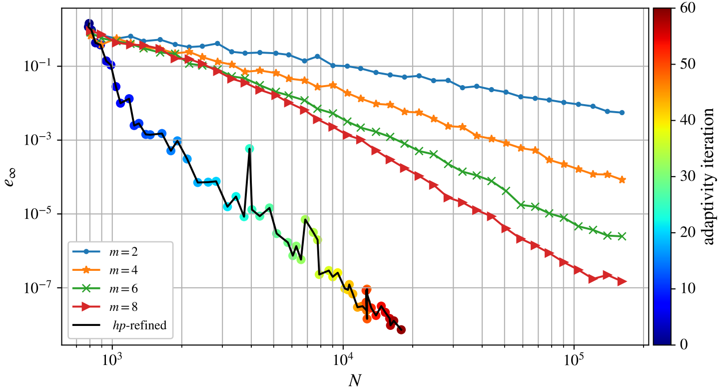
$$h_i^{new}(\boldsymbol{p}) = \frac{h_i^{old}}{\frac{\beta\eta_{max} - \eta_i}{\beta\eta_{max} - \eta_{min}}\left(\frac{1}{\vartheta} - 1\right) + 1}$$



hp-adaptivity: Poisson problem



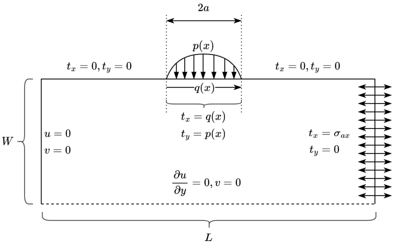
hp-adaptivity: Convergence Rates



hp-adaptivity: Fretting Fatigue Problem

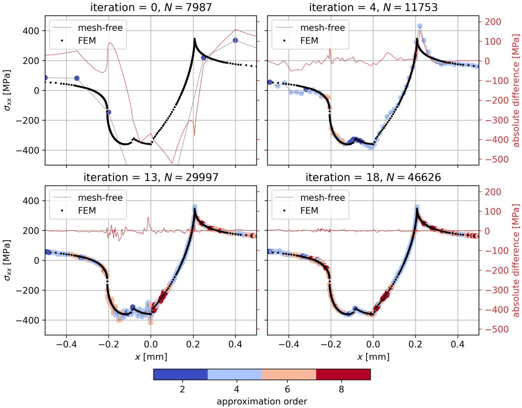
The problem is governed by the Cauchy-Navier equations

$$(\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu \nabla^2 \boldsymbol{u} = \boldsymbol{f}$$



Setup

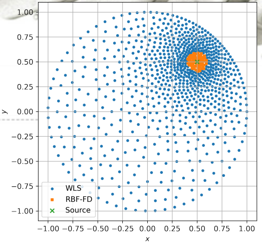
RBF-FD
PHS order $k = 3$
Monomial augmentation with $m \in \{2, 4, 6, 8\}$
IMEX with monomials $m \in \{4, 6, 8, 10\}$

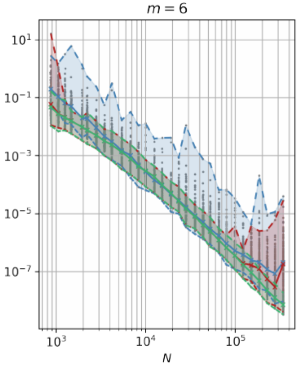


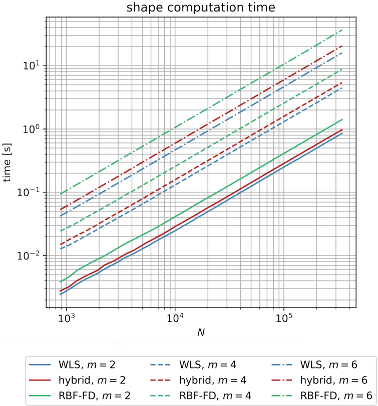
- Good agreement with FEM solution

Hybrid WLS–RBF-FD approximation

- Spatially-variable approximation method
- For greater solving efficiency: RBF-FD should be employed on as little nodes as possible







Hybrid scattered-regular

Discretization:

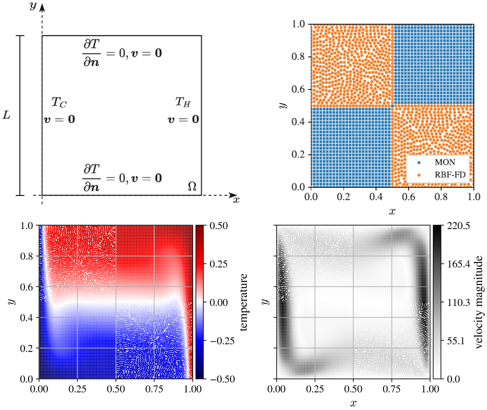
- Scattered nodes only where necessary
- Regular nodes elsewhere

Approximation:

- RBF-FD on scattered nodes ($n = 12$ in 2D for second order approximation)
- MON on regular nodes ($n = 5$ in 2D)

Note:

No special treatment required on the transition from scattered to regular nodes.



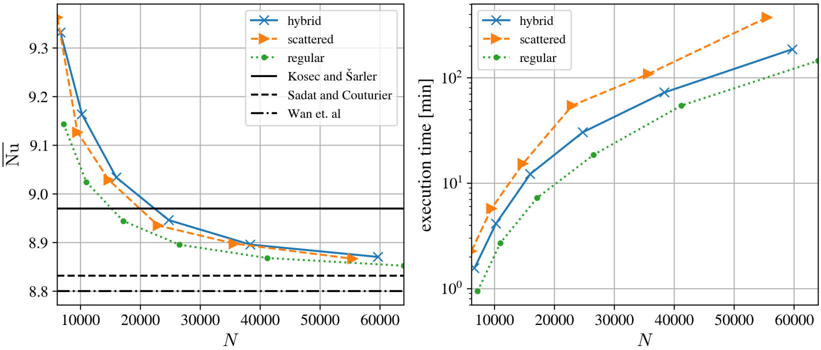
$$\nabla \cdot \boldsymbol{v} = 0,$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla \cdot (\nabla \boldsymbol{v}) - \boldsymbol{g} T_\Delta,$$

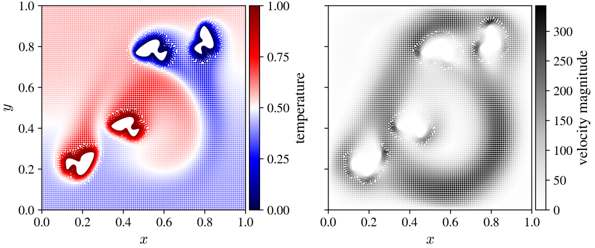
$$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T = \frac{1}{\mathrm{RePr}} \nabla \cdot (\nabla T),$$

Hybrid scattered-regular: DVD convergence

Nusselt number $\mathrm{Nu} = \frac{L}{T_H - T_C} \frac{\partial T}{\partial \boldsymbol{n}}$: the ratio between convective and conductive heat transfer (here computed along the cold wall).



Hybrid scattered-regular: Irregular domains

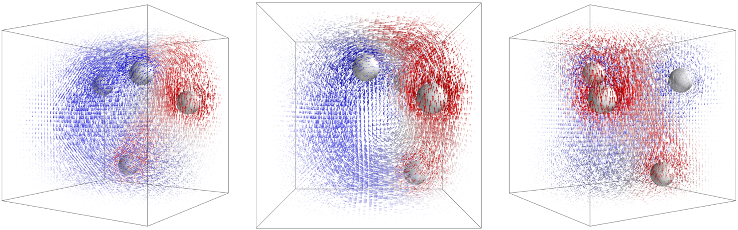


2D:

Approximation	$\overline{\text{Nu}}$	execution time [min]	N
scattered	12.32	46.31	10 534
hybrid	12.36	29.11	11 535

3D:

Approximation	$\overline{\text{Nu}}$	execution time [h]	N
scattered	7.36	48.12	65 526
hybrid	6.91	20.54	74 137



Summary

Presented:

- Monomial augmentation guidelines
- *p*-refinement
- *hp*-adaptive solution procedure
- Hybrid WLS–RBF-FD method
- Hybrid scattered-uniform discretization approach

Future work:

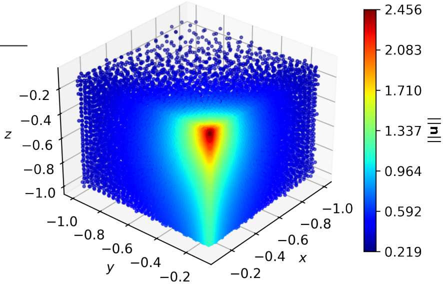
- Different marking and refinement strategies employed by the *hp*-adaptive solution procedure
- Different error indicators in the *hp*-adaptivity
- *hp*-adaptivity in the context of fluid flow problems

Questions?

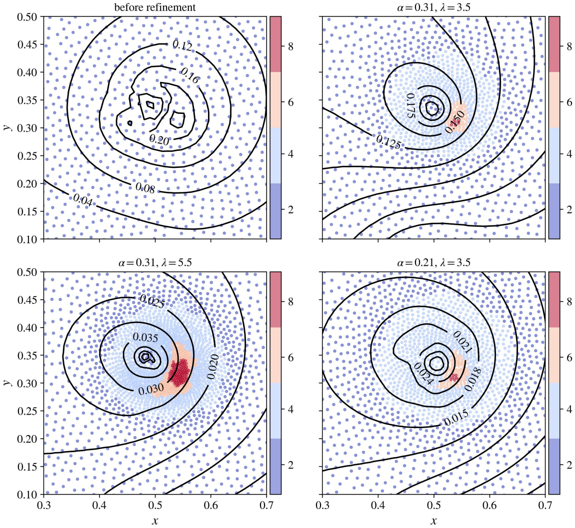
Hybrid WLS–RBF-FD approximation: 3D-Domain

Approximation	e_∞	t_{shape} [s]	$N_{\text{RBF-FD}}/N \cdot 100$
WLS	NaN	4.74	0.00
RBF-FD	$9.48 \cdot 10^{-5}$	8.22	100.00
hybrid	$2.37 \cdot 10^{-3}$	6.15	34.28

- RBF-FD part improves stability
- Shorter execution times observed
- Other combination of approximation method could be used



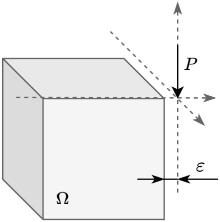
hp-adaptivity: Brief Study of Free Parameters



hp-adaptivity: Boussinesq's Problem

The problem is governed by the Cauchy-Navier equations

$$(\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu\nabla^2\boldsymbol{u} = \boldsymbol{f}$$



- Good agreement with closed form solution
- + Avoided fine-tunning with free parameters

Setup

RBF-FD
PHS order $k = 3$
Monomial augmentation with $m \in \{2, 4, 6, 8\}$
IMEX with monomials $m \in \{4, 6, 8, 10\}$