hp-adaptive method for solving partial differential equations

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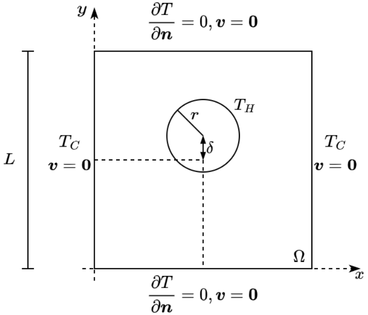
Motivation

Why numerical solution?

Realistic problems do not have closed form solutions.

$$\nabla \cdot \vec{v} = 0,$$
$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla p + \nabla \cdot (Ra \nabla \vec{v}) - \vec{g} Ra Pr T_\Delta,$$
$$\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = \nabla \cdot (\nabla T)$$



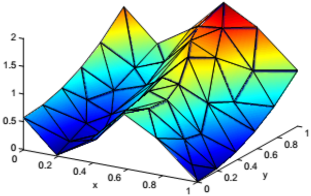
Numerical treatment is required:

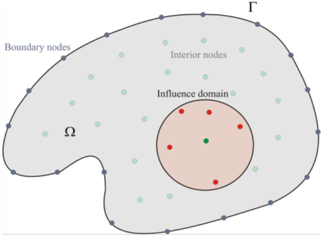
1. Domain discretization
2. Differential operator approximation
3. PDE discretization
4. Solve sparse linear system

Differential operator approximation

$$(\mathcal{L}u)(\boldsymbol{x}_c) \approx \sum_{i=1}^{n} w_i u(\boldsymbol{x}_i)$$

$$\mathcal{L}\Big|_{\boldsymbol{x}_c} = \boldsymbol{w}_{\mathcal{L}}(\boldsymbol{x}_c)^T$$





Approximation basis

RBF-FD

- Polyharmonic splines augmented with monomials
- + Higher stability
- - Computationally demanding

- Can operate on scattered nodes
- Control over the approximation order

Polyharmonic splines

$$f(r) = \begin{cases} r^k, & k \text{ odd} \\ r^k \log r, & k \text{ even} \end{cases},$$

Augmentation with $N_p = \binom{m+d}{m}$ monomials p with orders up to and including degree m,

$$\begin{bmatrix} \boldsymbol{F} & \boldsymbol{P} \\ \boldsymbol{P}^\mathsf{T} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\ell}_f \\ \boldsymbol{\ell}_p \end{bmatrix}.$$

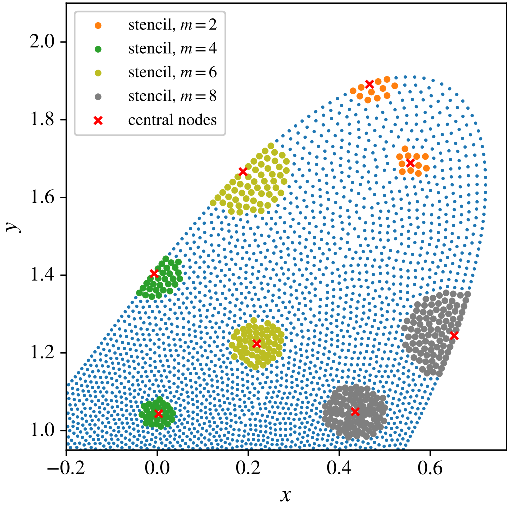
Stencil size

For a stable RBF-FD approximation, a recommended support size is

$$n = 2\binom{m+d}{d}.$$

m	$d=1$	$d=2$	$d=3$
2	6	12	20
4	10	30	70
6	14	56	168
8	18	90	330

Table 1: Support sizes in different domain dimensions d for various augmentation orders m.

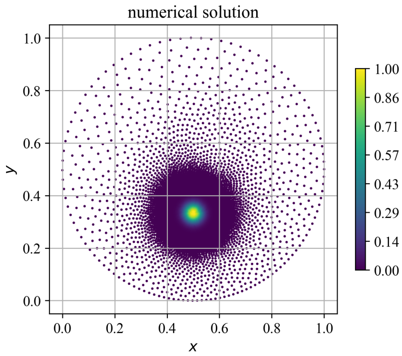


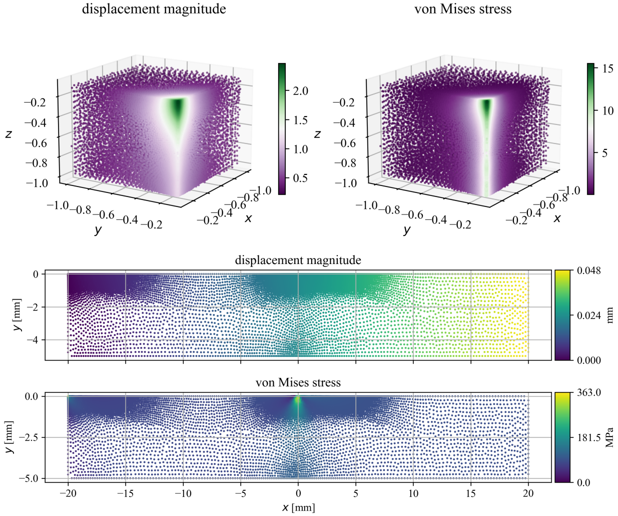
hp-adaptive solution procedure

Problems of interest

Mostly problems with:

- Strong sources
- Singularities

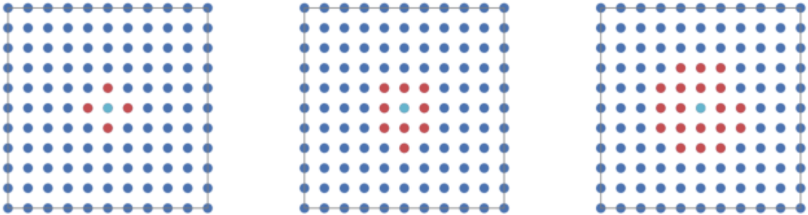




Refinement methods

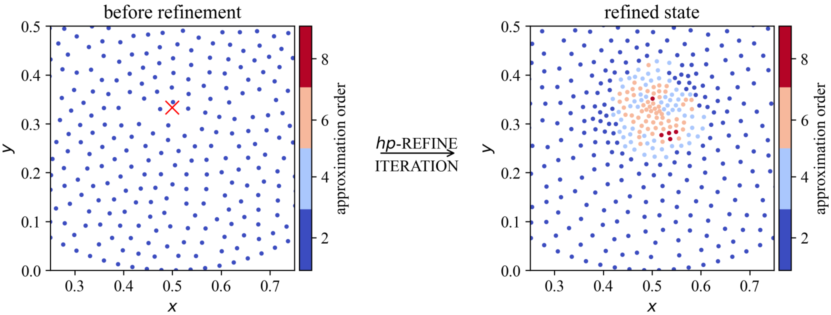
Refining methods are indispensable in problems where the solution error varies significantly throughout the computational domain.
Refinement gravely improves accuracy of numerical method.

- p-refinement
- h-refinement
- hp-refinement



hp-adaptivity: Workflow

1. **Solve** – A numerical solution $\widehat{u}$ is obtained.
2. **Estimate** – An estimate of the spatial accuracy of the numerical solution is calculated using error indicators.
3. **Mark** – Depending on the error indicator values η_i, a marking strategy is used to mark the computational nodes for (de)refinement.
4. **Refine** – Refinement strategy is employed to define the amount of the (de)refinement.



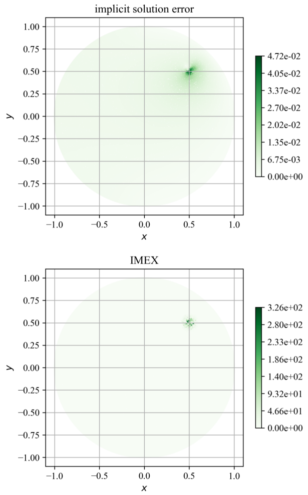
hp-adaptivity: Estimate module – IMEX

Consider a problem of type

$$\mathcal{L}u = f_{RHS}.$$

The IMplicit-EXplicit error indicator:

1. Obtain implicit solution $u^{(im)}$ to governing problem using low-order approximations of $\mathcal{L}$, i.e. $\mathcal{L}^{(lo)}_{(im)}$.
2. Obtain high-order approximations of explicit operators $\mathcal{L}$, i.e. $\mathcal{L}^{(hi)}_{(ex)}$
3. Apply $\mathcal{L}^{(hi)}_{(ex)}$ to $u^{(im)}$ and obtain $f_{(ex)}$ in the process
4. Compare f_{RHS} and $f_{(ex)}$



hp-adaptivity: Mark module

The modified Texas Three-Fold strategy for error indicator field η

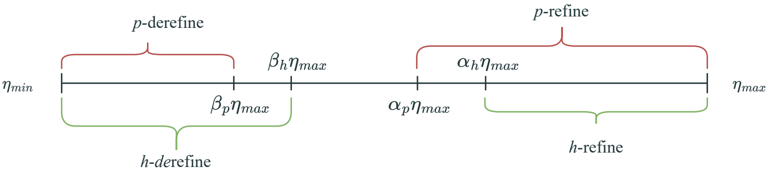
$$\begin{cases} \eta_i > \alpha \eta_{max}, & \text{refine} \\ \beta \eta_{max} \leq \eta_i \leq \alpha \eta_{max}, & \text{do nothing} \\ \eta_i < \beta \eta_{max}, & \text{derefine} \end{cases}.$$

Advantage

Easy to understand and implement.

Problem

Does not lead to optimal results.



hp-adaptivity: Refine module

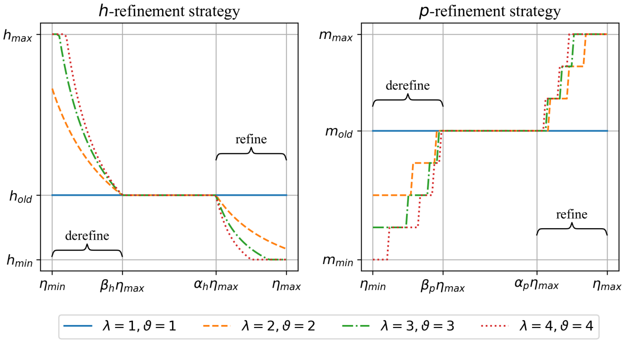
Defining the amount of (de)refinement.

h-refine:

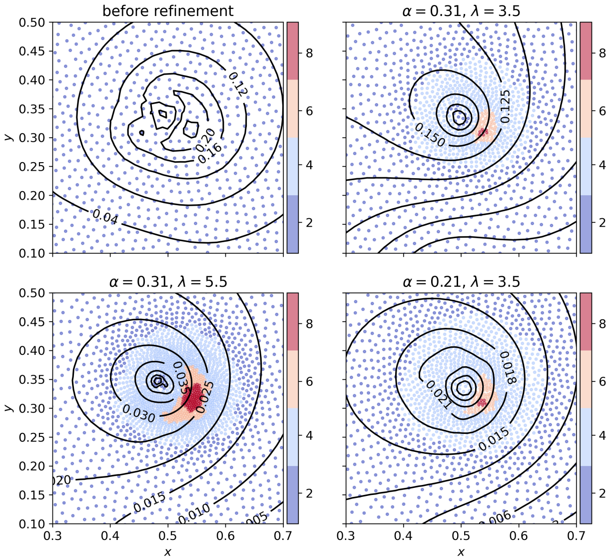
$$h_i^{new}(\boldsymbol{p}) = \frac{h_i^{old}}{\frac{\eta_i - \alpha\eta_{max}}{\eta_{max} - \alpha\eta_{max}}(\lambda - 1) + 1}$$

h-derefine:

$$h_i^{new}(\boldsymbol{p}) = \frac{h_i^{old}}{\frac{\beta\eta_{max} - \eta_i}{\beta\eta_{max} - \eta_{min}}\left(\frac{1}{\vartheta} - 1\right) + 1},$$



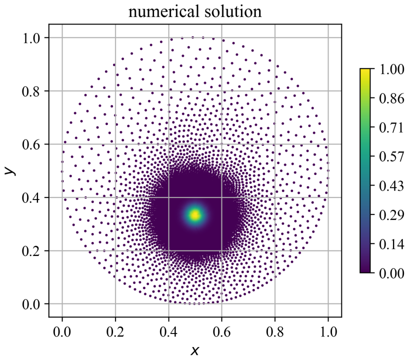
hp-adaptivity: Brief study of free parameters



hp-adaptivity: Example problem

Poisson problem with exponentially strong source in the domain

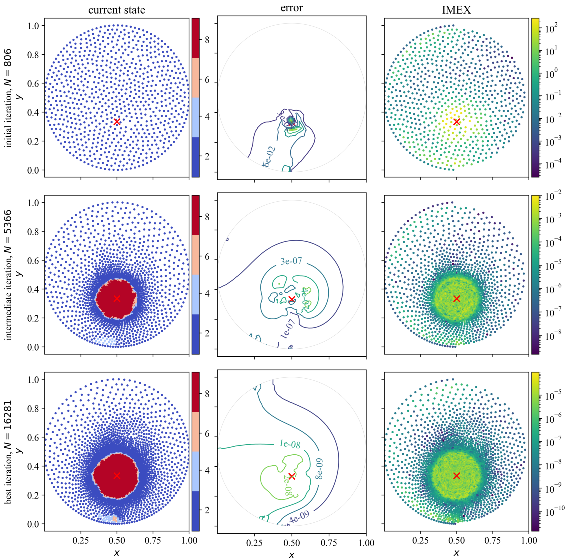
$$
\begin{aligned}
\nabla^2 u(\boldsymbol{x}) &= 2ae^{-a\|\boldsymbol{x}-\boldsymbol{x}_s\|^2}(2a\|\boldsymbol{x}-\boldsymbol{x}_s\| - d) && \text{in } \Omega, \\
u(\boldsymbol{x}) &= e^{-a\|\boldsymbol{x}-\boldsymbol{x}_s\|^2} && \text{on } \Gamma_d, \\
\nabla u(\boldsymbol{x}) &= -2a(\boldsymbol{x}-\boldsymbol{x}_s)e^{-a\|\boldsymbol{x}-\boldsymbol{x}_s\|^2} && \text{on } \Gamma_n
\end{aligned}
$$



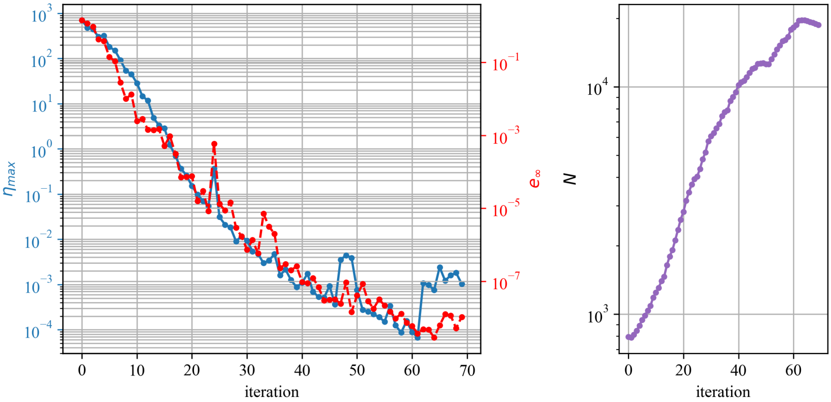
Setup

- RBF-FD
- PHS order $k = 3$
- Monomial augmentation with $m \in \{2, 4, 6, 8\}$
- IMEX with monomials $m \in \{4, 6, 8, 10\}$

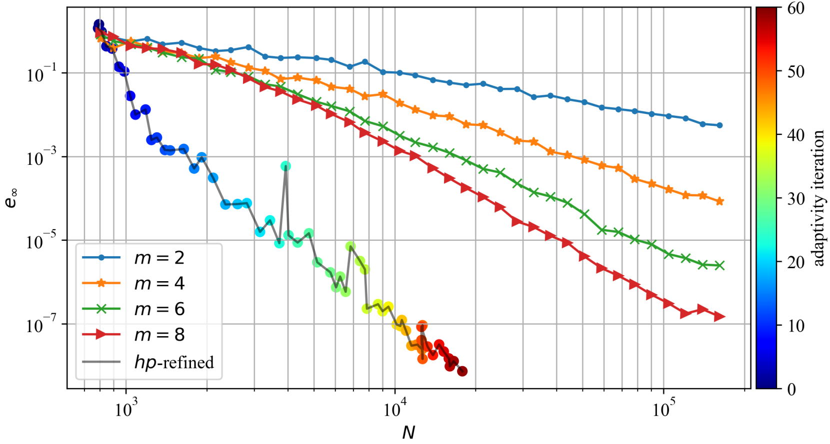
hp-adaptivity: Demonstration



hp-adaptivity: Convergence rates – IMEX

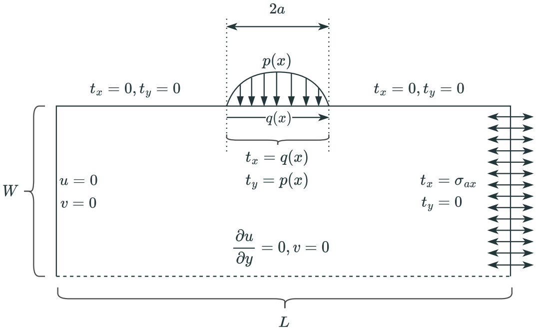


hp-adaptivity: Convergence rates – comparrison



The problem is governed by the Cauchy-Navier equations

$$(\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu\nabla^2\boldsymbol{u} = \boldsymbol{f}$$



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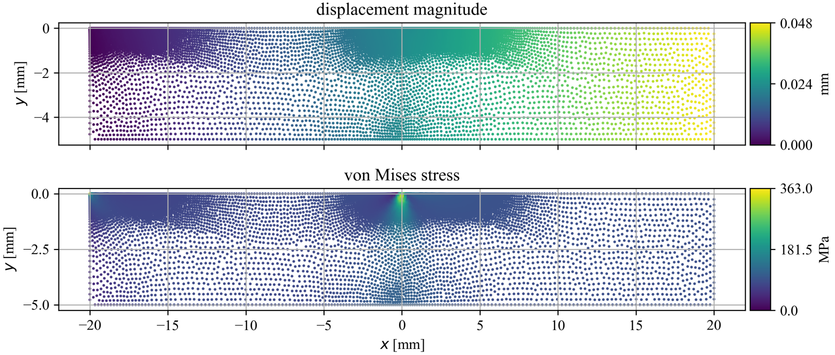
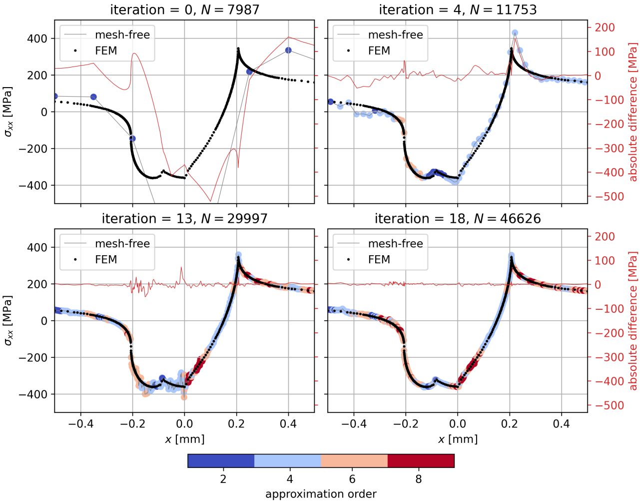
hp-adaptivity: Fretting fatigue problem – Example solution



hp-adaptivity: Fretting fatigue problem – Surface traction

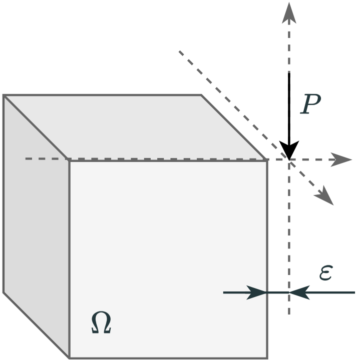
- Surface traction σ_{xx} under the contact
- Non-trivial local approximation order distribution
- Increased nodal density
- Good agreement with FEM solution



hp-adaptivity: Boussinesq's problem

The problem is governed by the Cauchy-Navier equations

$$(\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu\nabla^2\boldsymbol{u} = \boldsymbol{f}.$$



Setup

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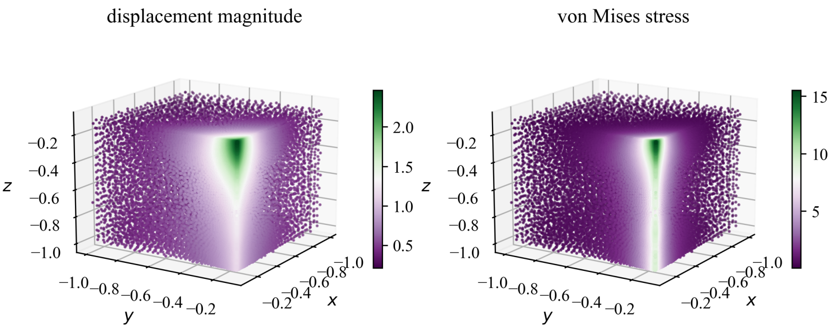
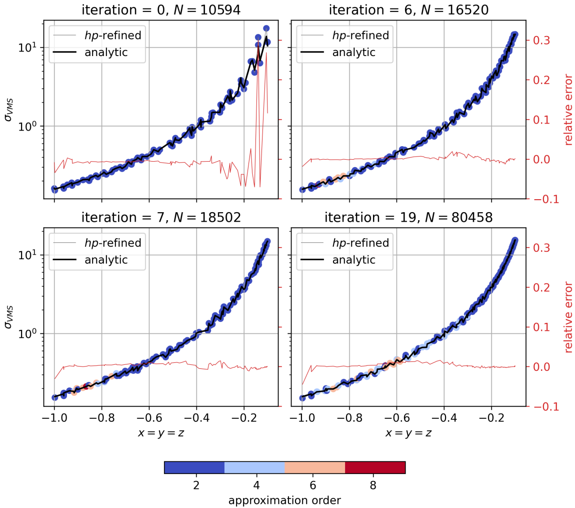
hp-adaptivity: Boussinesq problem – Example solution



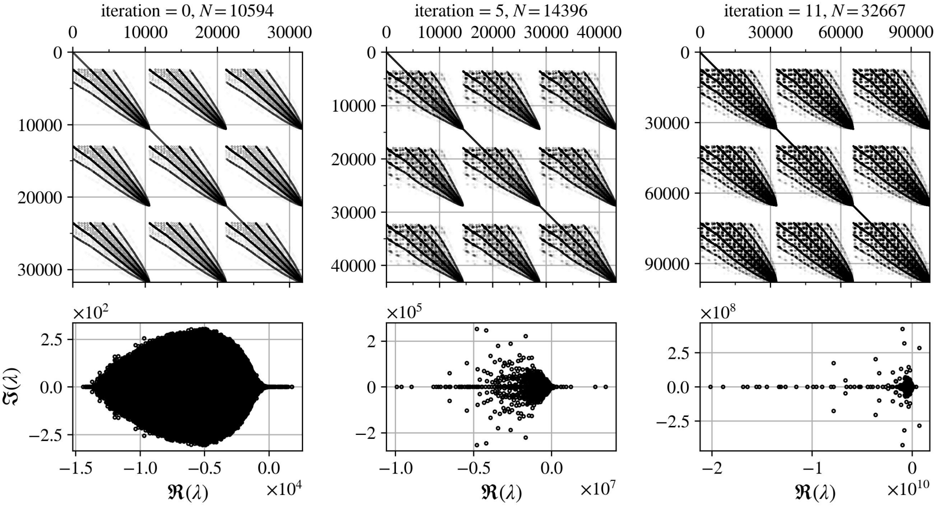
hp-adaptivity: Boussinesq problem – von Mises stress

- The von Mises stress σ_{VMS} along body diagonal
- Non-trivial local approximation order distribution
- Increased nodal density
- Good agreement with closed form solution

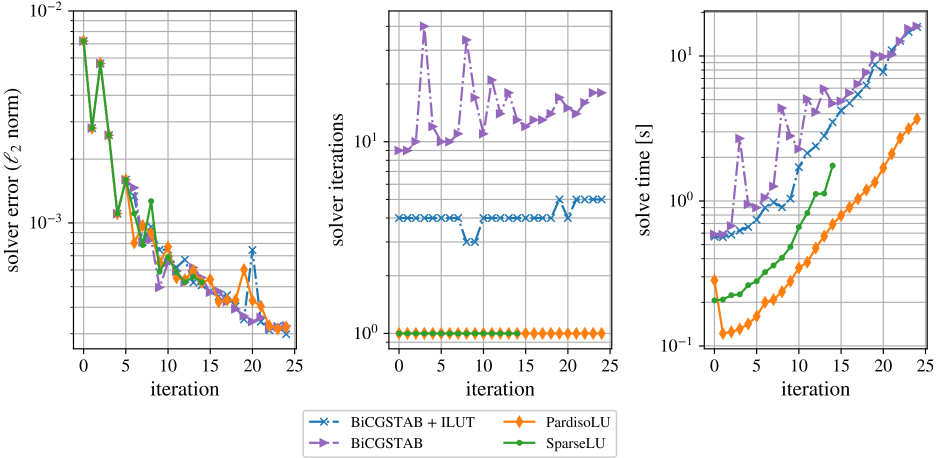
\+ Avoided fine-tunning with free parameters



hp-adaptivity: Boussinesq problem – Solvers analysis



hp-adaptivity: Boussinesq problem – Matrix spectrum



Conclusions

Summary

Presented:

- Adaptive *hp*-refinement

Future work:

⋆ Different marking and refinement strategies in the *hp*-adaptivity

⋆ Different error indicators in the *hp*-adaptivity

⋆ *hp*-adaptivity in the context of fluid flow problems

Questions?